HIMALAYAN JOURNAL of Basic and Applied Sciences

REVIEW ARTICLE

Journal Website: https://himalayanjournal.org/
Email: editor@orchidsocietyofassam.com

Himalayan Journal of Basic and Applied Sciences
vol.1, Issue 2, Dec., 2025; Page No. 44-47
Doi:https://doi.org/10.5281/zenodo.18181644
ISSN NO. Applied For (Print edition) ISSN NO. **3107-9113** (online edition)

A Very Short Note on Some of the Fuzzy Operations

Bikram Baro[1*] and Sahalad Borgoyary[2]

[1*]Research Scholar, Department of Mathematics, Central Institute of Technology, Kokrajhar, Assam (India)
[2]Department of Mathematics, Central Institute of Technology, Kokrajhar, Assam (India)

Corresponding Author: barobikram@gmail.com

Abstract

This article reviews commonly used operations of fuzzy sets and compares them with classical (crisp) set operations. It highlights similarities, differences, and analogues through clear definitions, theorems, and illustrative examples. Special aspects of various fuzzy set operations are discussed, including Baruah's newly proposed definitions. Rather than proposing a new theory, the article offers conceptual insights into operational relationships between fuzzy and crisp sets.

Keywords: *Fuzzy sets, fuzzy operations, Boolean algebra, classical set.*

Introduction

The four fundamental algebraic operations are addition, subtraction, multiplication, and division. In fuzzy set theory, analogous operations include complement, union, intersection, algebraic sum, and algebraic product. This paper presents an overview of well-established applications of fuzzy operations widely used by researchers. Zadeh (1975) introduced additional operations such as bounded sum and bounded difference, which expanded the operational framework of fuzzy sets. Defuzzification is an important process in which a fuzzy output is converted into a single crisp value, generally using the same membership function employed during fuzzification. Common defuzzification techniques include the centre of area (COA), centre of maxima (COM), mean of maxima (MOM), centre of sums (COS), and weighted average (WA) methods. Fuzzy sets also form a unitary commutative semiring under union and intersection operations (Mizumoto, 1981). This study examines fuzzy set operations that satisfy certain algebraic laws and those that violate classical Boolean principles, based on Zadeh's definitions and later extensions by a study (Baruah, 2013).

Definitions

The fuzzy sets were first introduced by Lotfi Asker Zadeh in 1965 as an extension of crisp set or non-fuzzy set. The fuzzy set can be defined as follows:

If X is the universal set, a fuzzy set A in X is defined by a membership function denoted by μ_A; that is $\mu_A: X \longrightarrow [0,1]$.

The standard fuzzy operations are fuzzy complement, intersection and union. These are defined respectively as:

$\bar{A}(x) = 1 - A(x)$; $(A \cap B)(x) = \min(A(x), B(x))$ and $(A \cup B)(x) = \max(A(x), B(x))$ for all $x \in X$.

For a fuzzy set *A* in *X* characterized by a membership function $f_A(x)$ for all *x* in *X,* Zadeh defined the product, algebraic sum and absolute difference as follows:

Received: 25th November, 2025; Revised & Accepted: 29th November, 2025; Published *Online: 5th December, 2025*
Published by Orchid Society of Assam

I. *Algebraic product:* The algebraic product of A and B is denoted by AB and is defined in terms of the membership functions of A and B by the relation

$$f_{AB} = f_A f_B$$

II. *Algebraic Sum:* The algebraic sum of A and B is denoted by A + B and is defined by

$$f_{A+B} = f_A + f_B - f_A f_B$$

III. *Absolute difference:* The absolute difference of A and B is denoted by $|A - B|$ and is defined by

$$f_{|A-B|} = |f_A - f_B|$$

IV. *Law of Contradiction and Law of Excluded Middle:* In fuzzy set theory, proposed by Zadeh, the following two laws does not hold good.

$A \cap A^c = \emptyset, AUA^c = X$ where, $\emptyset$ and X are empty and universal set respectively.

To verify the above, one has to show that $\min[A(x), A^c(x)] = 0$ and $\max[A(x), A^c(x)] = 1$ respectively. Obviously, the above equations violated for any value, $A(x) \in (0, 1)$. For example, let us consider a fuzzy set A on $X = \{0, 1, 2, 3\}$ is $A = \frac{0.3}{0} + \frac{0.6}{1} + \frac{0.1}{2} + \frac{1.0}{3}$. Then, $A^c = \frac{0.7}{0} + \frac{0.4}{1} + \frac{0.9}{2} + \frac{0.0}{3}$

Now, $\max[A(x), A^c(x)] = (A \cup A^c)(x) = \frac{0.7}{0} + \frac{0.6}{1} + \frac{0.9}{2} + \frac{1.0}{3} \neq 1, \forall\, x \in X$

$\min[A(x), A^c(x)] = (A \cap A^c)(x) = \frac{0.3}{0} + \frac{0.4}{1} + \frac{0.1}{2} + \frac{0.0}{3} \neq 0, \forall\, x \in X$. Therefore, the above two laws are violated.

V. *Fuzzy Complement:*

According to a study (Baruah H.K, 2013), a fuzzy set $A = \{x, \mu(x), x \in \Omega\}$ defined in the way as $A = \{x, 1, \mu(x), x \in \Omega\}$, so that its complement becomes $A^c = \{x, 1, \mu(x), x \in \Omega\}$. From this definition, it is clear that the result obtained by using the existing definition of complement does not logically satisfy. Various important properties are used by all fuzzy complements. These relates the equilibrium of a fuzzy complement *c*, which is defined as any value *a* for which $c(a) = 1 - a$.

Theorem 1.1. Every fuzzy complement has at most one equilibrium.

Proof: Let c be an arbitrary fuzzy complement. An equilibrium of c is a solution of the equation $c(t) - t = 0$, $x \in [0, 1]$. We can prove this theorem by demonstrating that any equation $c(t) - t = a$, where *a* is a real constant, has at most one solution.

Now, let us assume that t_1 and t_2 are two distinct solutions of the equation $c(t) - t = a$ such that $t_1 < t_2$. Then, $c(t_1) - t_1 = a$ and $c(t_2) - t_2 = a$; so that we get,

$c(t_1) - t_1 = c(t_2) - t_2$ ………. (i). Since c is monotonic decreasing, $c(t_1) \geq c(t_2)$, and because of $t_1 < t_2$,

$c(t_1) - t_1 > c(t_2) - t_2$, which contradicts to (i). Hence, the equation $c(t) - t = a$ must have at most one solution.

The algebraic properties of fuzzy sets under the operations of bounded-sum, bounded-difference and bounded-product which were discussed by Zadeh in 1975 is discussed below.

VI. *Bounded-Sum:* Let A and B be two fuzzy sets. Then the bounded-sum of A and B is defined as

$A \oplus B \Leftrightarrow \mu_{A \oplus B} = 1 \wedge (\mu_A + \mu_B)$, where the operation $\wedge$ represents minimum.

From the above definition, we see that $A \oplus B = B \oplus A$

$(A \oplus B) \oplus C = A \oplus (B \oplus C), A \oplus \emptyset = A, A \oplus U = U, A \oplus \bar{A} = U$

VII. *Bounded-differences:* Let A and B be two fuzzy sets. Then the bounded-sum of A and B is defined as

$A \ominus B \Leftrightarrow \mu_{A \ominus B} = 0 \vee (\mu_A - \mu_B)$, where the operation $\vee$ represents maximum.

From the above definition we have,

$A \ominus B \neq B \ominus A$, $(A \ominus B) \ominus C \subseteq A \ominus (B \ominus C)$, $A \ominus (A \ominus B) = A \cap B$, $A \ominus \emptyset = A$, $\emptyset \ominus A = \emptyset$, $A \ominus U = \emptyset, U \ominus A = \bar{A}$

VIII. *Bounded-Product:* Let A and B be two fuzzy sets. Then the bounded-product of A and B is defined as A $\odot$ B $\Leftrightarrow \mu_{A \odot B} = 0 \vee (\mu_A + \mu_B - 1)$
From the above definition, we see that
$A \odot B = B \odot A, (A \odot B) \odot C = A \odot (B \odot C), A \odot \emptyset = \emptyset, A \odot U = A, A \odot \bar{A} = \emptyset$
IX. *Absolute difference:* For two fuzzy sets A and B, their absolute difference is defined as $f_{|A-B|} = |f_A - f_B|$, and is denoted by $|A - B|$
Example 1. Let us consider two fuzzy sets given by $A = \{\frac{0.4}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.7}{4}\}$ and $B = \{\frac{0.3}{1} + \frac{0.4}{2} + \frac{0.4}{3} + \frac{0}{4}\}$
Here, we are going to find out algebraic sum, algebraic product, bounded sum and bounded product of the given two fuzzy sets. From the above definition we have, *algebraic sum* of the fuzzy sets A and B,

$f_{A+B} = f_A + f_B - -f_A f_B = \{\frac{0.58}{1} + \frac{0.7}{2} + \frac{0.76}{3} + \frac{0.7}{4}\}$ then *algebraic product* of the fuzzy sets A and B, $f_{AB} = f_A f_B = \{\frac{0.12}{1} + \frac{0.2}{2} + \frac{0.24}{3} + \frac{0}{4}\}$

bounded-sum of the fuzzy sets A and B, $A \oplus B \Leftrightarrow \mu_{A \oplus B} = 1 \wedge (\mu_A + \mu_B) = \min[1, (\mu_A + \mu_B)] = \{\frac{0.7}{1} + \frac{0.9}{2} + \frac{1}{3} + \frac{0.7}{4}\}$

bounded-difference of two fuzzy sets A and B, $A \ominus B \Leftrightarrow \mu_{A \ominus B} = 0 \vee (\mu_A - \mu_B) = \{\frac{0.1}{1} + \frac{0.1}{2} + \frac{0.2}{3} + \frac{0.7}{4}\}$.

Various Types of Operations on Fuzzy Sets

Apart from operations of union and intersection, some important combinations of fuzzy set are discussed (Zadeh,1965). *Algebraic difference* of the fuzzy sets A(x) and B(x), for all $x \in X$, is denoted by $A(x) - B(x)$, which is defined as $A(x) - B(x) - \{(x, \mu_{A-B}(x), x \in X)\}$, where $\mu_{A-B}(x) = \mu_{A \cap B}(x)$. Therefore, if $A(x) = \{(1, 0.1), (2, 0.5), (3, 0.6), (4, 0.7)\}$, $B(x) = \{(1, 0.3), (2, 0.6), (3, 0.2), (4, 0.1)\}$
then, $\bar{B}(x) = \{(1, 0.7), (2, 0.4), (3, 0.8), (4, 0.9)\}$ so that, $A(x) - B(x) = \{(1, 0.1), (2, 0.4), (3, 0.6), (4, 0.7)\}$

The disjunctive sum of two fuzzy sets is also a very important operation in fuzzy set theory. If A and B be two fuzzy set then, $\bar{A}(x) = 1 - A(x)$, $\bar{B}(x) = 1 - B(x)$, $(A \cap \bar{B})(x) = \min(A(x), 1 - B(x))$ and $(\bar{A} \cap B)(x) = \min(1 - A(x), B(x))$. Then, the disjunctive sum of two fuzzy sets A and B, is defined by $(A \oplus B)(x) = \max\{\min(A(x), 1 - B(x)), \min(1 - A(x), B(x)\}$. For example, if $A = \{(t_1, 0.3), (t_2, 0.5), (t_3, 0.7), (t_4, 1.0)\}$ and $B = \{(t_1, 0.6), (t_2, 0.2), (t_3, 0.0), (t_4, 0.3)\}$, then
$\bar{A} = \{(t_1, 0.7), (t_2, 0.5), (t_3, 0.3), (t_4, 0.0)\}$ and $\bar{B} = \{(t_1, 0.4), (t_2, 0.8), (t_3, 1.0), (t_4, 0.7)\}$

Theorem 1.2 The standard fuzzy union is the only idempotent t- conorm.

Proof. We know that, $\max(a, a) = a$ for all $a \in [0, 1]$. Let us assume that there exists a *t*- conorm such that $i(a, a) = a$ for all $a \in [0, 1]$. Then, for any $a, b \in [0, 1]$, we have the following two cases:

Case I: $a \geq b$: If $a \geq b$, then $a = i(a, a) \geq i(a, b) \geq i(a, 0) = a$ i.e., $a \geq i(a, b) \geq a$. Therefore, $i(a, b) = a = \max(a, b)$

Case II: $a \leq b$: If $a \leq b$, then we get $b = i(b, b) \geq i(a, b) \geq i(0, b) = b$ i.e., $b \geq i(a, b) \geq b$. Hence, $i(a, b) = \max(a, b)$ for all $a, b \in [0, 1]$. Consequently, from the both cases we have, $i(a, b) = \max(a, b)$ for all $a, b \in [0, 1]$. In the similar way, we can also show that the standard fuzzy intersection is the only idempotent *t*- norm. One of the most commonly used operations on fuzzy sets is Cartesian product of fuzzy sets. We all know that the Cartesian product of two non-empty sets A and B is defined as $A \times B = \{(a, b): a \in A, b \in B\}$. In general, the Cartesian product of a family of sets $\{A_1, A_2, \dots, A_n\}$ is the set of all n-tuples $\langle a_1, a_2, \dots, a_n \rangle$ such that, $a_i \in A_i$ $(i = 1, 2, \dots, n)$, written as $A_1 \times A_2 \times \dots \times A_n = \{\langle a_1, a_2, \dots, a_n \rangle : a_i \in A_i$ for every $i = 1, 2, \dots, n\}$. If $A \neq B$ and A, B are non-empty, then $A \times B \neq B \times A$. Now, let us assume that $A_1(x), A_2(x), \dots, A_n(x)$ are membership functions of $A_1, A_2, \dots, A_n$ for all $x_1 \in A_1, x_2 \in A_2, \dots, x_n \in A_n$. Then the Cartesian product of the fuzzy sets $A_1, A_2, \dots, A_n$ is the *n*-tuple $(x_1, x_2, \dots, x_n)$ which is expressed as $(A_1 \times A_2 \times \dots \times A_n)(x_1, x_2, \dots, x_n) = \min[A_1(x), A_2(x), \dots, A_n(x)]$.
For example, let us consider the set $X = \{1, 2, 3\}$ and two fuzzy sets $A = \{(1, 0.2), (2, 0.5), (3, 0.8)\}$ and $B = \{(1, 0.7), (2, 0.4), (4, 0.1)\}$, then the cartesian product of fuzzy sets *A* and *B* is expressed as follows: $A \times B = \{[(1, 1), 0.2], [(1, 2), 0.2], [(1, 4), 0.1], [(2, 1), 0.5], [(2, 2), 0.4], [(2, 4), 0.1], [(3, 1), 0.7], [(3, 2), 0.4],$

$[(3,4),0.1]$. Now, $B \times A = \{[(1,1),0.2],[(1,2),0.5],[(1,3),0.7],[(2,1),0.2],[(2,2),0.4],[(2,3),0.4],$ $[(4,1),0.1],[(4,2),0.1],[(4,3),0.1]\}$.

Therefore, from the above also we have seen that for two fuzzy sets A and B, $A \times B \neq B \times A$, if $A \neq B$. Apart from these, some of the most operations on fuzzy sets are- defuzzification and ranking of fuzzy numbers.

Some Combinations of Operations on Fuzzy Sets

The De Morgan laws for fuzzy sets are: $c\big(i(a,b)\big) = u(c(a),c(b))$, $c\big(u(a,b)\big) = i(c(a),c(b))$, where, c be the fuzzy complement, *i* be the *t*-norm and u be the *t*-conorm. If *i* and u be the dual with respect to the fuzzy complement c, then it is called dual triple and is denoted by $\langle i,u,c\rangle$. Although, the law of excluded middle and the law of contradiction do not hold good in fuzzy sets theory, the fuzzy operations *i, u, c* satisfies the law of excluded middle and the law of contradiction.

Theorem 1.3 If $\langle i,u,c\rangle$, be a dual triple, then the fuzzy operations i, u, c satisfies the law of excluded middle and the law of contradiction.

Proof. From the First Characteristic Theorem of Fuzzy Complements, we have

$c(a) = g^{-1}(g(1) - g(a))$, where $c: [0,1] \longrightarrow [0,1]$ and $g: [0,1] \longrightarrow R$

Now, from First Characteristic Theorem of *t*-norm, we have $i(a,b) = g^{(-1)}(g(a) + g(b) - g(1))$

Also, from First Characteristic Theorem of *t*-conorm, we have $u(a,b) = g^{(-1)}(g(a) + g(b))$.

Therefore, $u\big(a,c(a)\big) = g^{(-1)}(g(a) + g(c(b))) = g^{(-1)}(g(a) + g(g^{-1}(g(1) - g(a))))$

$= g^{(-1)}(g(a) + g(1) - g(a)) = g^{(-1)}(g(1)) = 1$, for all $a \in [0,1]$. Hence, the law of excluded middle is satisfied.

Similarly, $i\big(a,c(a)\big) = g^{(-1)}(g(a) + g\big(c(a)\big) - g(1)) = g^{(-1)}(g(a) + g(g^{-1}(g(1) - g(a))) - g(1))$

$= g^{(-1)}(g(a) + g(1) - g(a) - g(1)) = g^{(-1)}(0) = 0$, for all $a \in [0,1]$

Hence, the law of contradiction is also satisfied. From the above theorem, the most important result arises that, if the dual triple $\langle i,u,c\rangle$ satisfies the law of excluded middle and the law of contradiction, then it does not satisfy the distributive law. Hence, the distributive law does not hold.

Conclusion

This article examines some fundamental operations on fuzzy sets and their relationship with classical set theory. While most classical set operations have natural analogues in fuzzy set theory, certain laws such as the law of excluded middle, the law of contradiction, and some laws of Boolean algebra are violated in fuzzy operations. Some studies state that fuzzy sets do not satisfy Boolean algebra, while others argue that fuzzy subsets partially follow its laws. However, it is widely accepted that the collection of all fuzzy subsets of a finite set cannot form a Boolean algebra, and that the laws of Boolean algebra often differ from those of real number algebra (Dhar M. 2011). Moreover, every special fuzzy Boolean algebra contains a special fuzzy Boolean subalgebra known as the improper subalgebra (Talukdar D., 2015). Since, for a fuzzy set A, neither $A \cap \bar{A}$ is the empty fuzzy set nor $A \cup \bar{A}$ is the universal fuzzy set, it can be concluded that fuzzy sets do not satisfy all the laws of Boolean algebra.

Declarations

Conflict of Interest: The authors declare that they have no conflict of interest

References

1. **Baruah H.K & Dhar M. (2013)**. Definition of the Complement of Fuzzy Numbers: An Exposition, *I.J. Intelligent Systems and Applications*, 8:73-82
2. **Dhar M., (2011).** Fuzzy Sets towards Forming Boolean Algebra, *International Journal of Energy, Information and Communications*, 2 (4): 137-142
3. **Mizumoto M. & Tanaka K., (1981).** Fuzzy Sets and Their Operations, *Information and Control*,48:30-48
4. **Talukdar Dwiraj & Rjbongshi S. K. (2015).** Some Aspects of Special Fuzzy Boolean Algebra, *International Conference on Frontiers in Mathematics*, 26-28.
5. **Zadeh L.A., (1965).** Fuzzy Sets, *Information and Control*, 8: 338-353